Exact sequences, lower central series and representations of surface braid groups

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Abstract

We consider exact sequences and lower central series of (surface) braid groups and we explain how they can prove to be useful for obtaining representations for surface braid groups. In particular, using a completely algebraic framework, we describe the notion of extension of a representation introduced and studied recently by An and Ko and independently by Blanchet.

1 Introduction

The faithful linear representations of Bigelow-Krammer-Lawrence of the Artin braid groups B_n are probably one of the most important recent discoveries in the theory of braid groups, and as such, have been intensively studied over the last few years. They have also been extended to other groups, such as Artin-Tits groups of spherical type (see for instance [14, 15]). However, with the exception of a few results [3, 11, 27], their generalisation in a more topological direction, to braid and mapping class groups of surfaces for example, as well as the linearity of these groups, are open problems in general.

The aim of this paper is twofold. The first is to underline the relevance of short exact sequences of braid groups and their generalisations to the study of representations of these groups. As we shall recall in Section 2, the Burau and Bigelow-Krammer-Lawrence representations appear when one studies certain 'mixed' extensions of B_n arising from fibrations at the level of configuration spaces. This extension splits, and in the simplest case, its kernel is the fundamental group of the n-punctured disc $\mathbb{D}_n$. The Burau representation then occurs as the induced action of B_n on the homology of the infinite cyclic covering $\widetilde{\mathbb{D}}_n$ of $\mathbb{D}_n$.

The Bigelow-Krammer-Lawrence representations may be obtained in a similar way in terms of the Borel-Moore middle homology group of a $\mathbb{Z}^2$-covering of $\mathbb{D}_n$ (see Section 2.3 for more details). Motivated by these constructions, one possible strategy for obtaining linear representations of (surface) braid groups is to study exact sequences of these groups. With this in mind, we recall the definitions of

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surface braid groups and their short exact sequences in Section 3, as well as the known results concerning the splitting of these sequences (Lemma 3.2). In the case of the short exact sequence of 'mixed' Artin braid groups, the induced short exact sequence on the level of commutator subgroups brings into play groups and homomorphisms that appear in the construction of Bigelow-Krammer-Lawrence representations of B_n (see the end of Section 2.3).

Before stating similar results for exact sequences of surface braid groups, in Section 4, we recall the basic definitions pertaining to the lower central series $(\Gamma_i(G))_{i\in\mathbb{N}}$ of a group G, and we show that if G is the braid or mixed braid group (with a sufficiently large number of strings) of a compact orientable surface $\widehat{\Sigma}_g$ with a single boundary component, the quotient group $G/\Gamma_3(G)$ is a semi-direct product of free Abelian groups. We provide group presentations for such quotients (Lemma 4.9 and Corollary 4.10) that will play an important rôle in the rest of the paper.

In [1], An and Ko described an extension of the Bigelow-Krammer-Lawrence representations of B_n to braid groups of orientable surfaces of positive genus and with non-empty boundary. However, it is not currently known whether these representations are faithful. The representation is based on the regular covering arising from a projection of the n-th braid group of a surface Σ with non-empty boundary onto a specific group G_Σ, constructed in a technical manner in order to satisfy certain homological constraints (Section 3 and Definition 2.2 of [1]) and which turns out to be a Heisenberg group; more recently, the above projection has been independently studied by Christian Blanchet to obtain a representation of a large subgroup of the Torelli group of a surface with one boundary component containing the Johnson subgroup [13].

The second aim of our paper is to show that technical construction proposed in [1] of the representations may be described in terms of lower central series and exact sequences of surface braid groups: this is the object of Section 5. As we mentioned above, in the case of Artin braid groups, the induced short exact sequence on the Γ_2-level gives rise to elements used in the construction of the Bigelow-Krammer-Lawrence representations. In the case of surface braid groups, the corresponding construction on the same level does not work (Proposition 5.4), but if we take this construction a stage further, to the Γ_3-level, we obtain the corresponding objects of the An-Ko representations. More precisely we show how to use the Γ_3-level to extend Bigelow-Krammer-Lawrence representations and we prove that such extensions are unique up to isomorphism (Propositions 5.7 and 5.8), according to Definition 5.2.

We finish the main part of the paper with another possible application of the lower central series of surface braid groups by showing that the standard length function on B_n admits a unique extension to a homomorphism whose source is the braid group of a surface of positive genus with one boundary component (Proposition 5.10), and that there is no such extension if the surface is closed and orientable (Proposition 5.11). In an Appendix, we discuss the relationship between the splitting of the 'mixed' surface braid group sequences and that of their restriction to the corresponding pure braid groups. In the terminology of [20], we

show that this restriction never give rises to an *almost-direct* product (an almost-direct product structure means that the extension is split, and that the induced action of the lower central series quotient $K/\Gamma_2(K)$ on the kernel K is trivial).

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2 Exact sequences, lower central series and representations of classical braid groups

2.1 Surface braid groups and configuration spaces

Surface braid groups are a natural generalisation of both the classical braid groups and of the fundamental group of surfaces. First defined by Zariski during the 1930's, they were re-discovered by Fox during the 1960's, and have been used subsequently in the study of mapping class groups.

We recall the definition due to Fox of these groups in terms of fundamental groups of configuration spaces [21]. Let Σ be a connected surface. Let $\mathbb{F}_n(\Sigma) = \Sigma^n \setminus \Delta$, where Δ is the set of n-tuples $x = (x_1, \ldots, x_n) \in \Sigma^n$ for which $x_i = x_j$ for some $i \neq j$. The fundamental group $\pi_1(\mathbb{F}_n(\Sigma))$ is called the *pure braid group* on n strands of the surface Σ; it shall be denoted by $P_n(\Sigma)$. There is a natural action of the symmetric group S_n on $\mathbb{F}_n(\Sigma)$ by permutation of coordinates; the fundamental group $\pi_1(\mathbb{F}_n(\Sigma)/S_n)$ is called the *braid group* on n strands of the surface Σ and shall be denoted by $B_n(\Sigma)$. Then $\mathbb{F}_n(\Sigma)$ is a regular $n!$-fold covering of $\mathbb{F}_n(\Sigma)/S_n$ that gives rise to the following short exact sequence:

$$1 \longrightarrow P_n(\Sigma) \longrightarrow B_n(\Sigma) \longrightarrow S_n \longrightarrow 1. \tag{1}$$

Regarded as a subgroup of S_{k+n}, the group $S_k \times S_n$ acts on $\mathbb{F}_{k+n}(\Sigma)$. The fundamental group $\pi_1\left(\mathbb{F}_{k+n}(\Sigma)/(S_k \times S_n)\right)$ will be called the *mixed braid group of* Σ *on* (k, n) *strands*, and shall be denoted by $B_{k,n}(\Sigma)$. Notice that $B_{k,n}(\Sigma)$ embeds canonically in $B_{k+n}(\Sigma)$. These intermediate groups between pure braid and braid groups of a surface, known as 'mixed' braid groups, play an important rôle in [1]. They were defined previously in [23, 31, 34], and were studied in more detail in [25] in the case where Σ is the 2-sphere $\mathbb{S}^2$.

2.2 Fibrations and induced exact sequences

We recall that $\pi_1(\mathbb{F}_n(\mathbb{D}^2))$ is isomorphic to the pure braid group on n strands, usually denoted by P_n, while $\pi_1(\mathbb{F}_n(\mathbb{D}^2)/S_n)$ is isomorphic to the braid group B_n on n strands. In what follows, the mixed braid group on (k, n) strands $\pi_1\left(\mathbb{F}_{k+n}(\mathbb{D}^2)/(S_k \times S_n)\right)$ will be denoted simply by $B_{k,n}$.

The fibration $\mathbb{F}_{k+n}(\mathbb{D}^2)/(S_k \times S_n) \longrightarrow \mathbb{F}_n(\mathbb{D}^2)/S_n$ given by forgetting the first k coordinates is a locally-trivial fibration whose fibre over a point $\{x_1, \ldots, x_n\}$ may be identified with the orbit space $\mathbb{F}_k(\mathbb{D}^2 \setminus \{x_1, \ldots, x_n\})/S_k$. From now on we denote $\mathbb{D}^2 \setminus \{x_1, \ldots, x_n\}$ by $\mathbb{D}_n$. Let us also denote $\pi_1(\mathbb{F}_k(\mathbb{D}_n)/S_k)$ by $B_k(\mathbb{D}_n)$; this group turns out to be isomorphic to the subgroup of B_{k+n} consisting of braids where the last n strands are trivial (vertical). The long exact sequence in homotopy of the above fibration yields a short exact sequence.

Lemma 2.1. *Let $k, n \in \mathbb{N}$. The Fadell-Neuwirth fibration $\mathbb{F}_{k+n}(\mathbb{D}^2) \longrightarrow \mathbb{F}_n(\mathbb{D}^2)$ induces the short exact sequence:*

$$1 \longrightarrow B_k(\mathbb{D}_n) \longrightarrow B_{k,n} \longrightarrow B_n \longrightarrow 1\,. \tag{MB}$$

In a similar way, we may obtain the more well-known short exact sequence of pure braid groups.

$$1 \longrightarrow P_k(\mathbb{D}_n) \longrightarrow P_{k+n} \longrightarrow P_n \longrightarrow 1. \tag{PB}$$

Here $P_k(\mathbb{D}_n)$ denotes the fundamental group of $\mathbb{F}_k(\mathbb{D}_n)$, which is isomorphic to the subgroup of P_{k+n} consisting of pure braids where the last n strands are vertical. Notice that the short exact sequences (MB) and (PB) split for all $k \geq 1$, where the section is given geometrically by adding k trivial strands 'at infinity' (see for instance [1, 6]).

2.3 Linear representations for the braid group B_n

When $k = 1$, the short exact sequence (PB) plays a central rôle in the study of Vassiliev invariants of braid groups and of Lie Algebras related to pure braid groups. We refer to [33] for the classical case and to [7, 16, 28] for analogous results in the case of surface braid groups.

In what follows, we will focus on the relevance of such short exact sequences to linear representations of braid groups and their topological generalisations. Let us start with the case $k = 1$. The group B_n may be interpreted as the mapping class group of $\mathbb{D}_n$ [12]. We thus obtain an action of B_n on $\mathbb{D}_n$ that induces an action on $\pi_1(\mathbb{D}_n)$, the latter being isomorphic to the free group F_n on n generators. This action, which is faithful, coincides with the action by conjugation of B_n on $B_1(\mathbb{D}_n)$ defined by the natural section of (MB). In this way, we recover the famous Artin representation of the braid group B_n as a subgroup of the group of automorphisms of F_n. Analogously, we have an action of P_n as the pure mapping class group of $\mathbb{D}_n$ on $P_1(\mathbb{D}_n) \simeq F_n$ what is faithful and coincides with the action by conjugation of P_n on $P_1(\mathbb{D}_n)$ defined by the natural section of (PB). Composing the Artin representation with the Magnus representation associated to the length function $p_1 : B_1(\mathbb{D}_n) \longrightarrow \mathbb{Z}$ (see for instance [2]) we obtain the (non reduced) Burau representation of B_n. In the case of the pure braid group, we obtain the Gassner representation of P_n ([2]) in a similar way.

The Burau representation also has a homological interpretation (see for instance Chapter 3 of [30]). Furthermore, it admits certain generalisations. Indeed,

for any $k \geq 1$ we may observe that B_n, regarded as the mapping class group of $\mathbb{D}_n$, acts on $\mathbb{F}_k(\mathbb{D}_n)/S_k$ and therefore on its fundamental group, $B_k(\mathbb{D}_n)$. The induced action of B_n on $B_k(\mathbb{D}_n)$ coincides with the action by conjugation of B_n on $B_k(\mathbb{D}_n)$ defined by the natural section of (MB). In order to look for (linear) representations, we consider regular coverings associated with normal subgroups of $B_k(\mathbb{D}_n)$, and we try to see if the induced action on homology is well defined. In other words, we wish to study surjections of $B_k(\mathbb{D}_n)$ onto a group G_k subject to certain constraints.

When $k = 1$, we consider as before the length function $p_1 : B_1(\mathbb{D}_n) \longrightarrow G_1 = \mathbb{Z} = \langle t \rangle$. Since the action of B_n on $B_1(\mathbb{D}_n)$ commutes with $p_1 : B_1(\mathbb{D}_n) \longrightarrow \mathbb{Z}$, B_n acts on the regular covering $\widetilde{\mathbb{D}}_n$ of $\mathbb{D}_n$. The induced action on the first homology group of $\widetilde{\mathbb{D}}_n$ is the (reduced) Burau representation of B_n.

For $k > 1$, let G_k be the group $\mathbb{Z}^2 = \langle q, t \rangle$. The corresponding morphism $p_k : B_k(\mathbb{D}_n) \longrightarrow G_k$ for $k > 1$ sends the classical braid generators $\sigma_1, \ldots, \sigma_k$ to q and the generators $\zeta_1, \ldots, \zeta_n$, corresponding to the generators of $\pi_1(\mathbb{D}_n)$, to t. Since the action of B_n on $B_k(\mathbb{D}_n)$ commutes with $p_k : B_k(\mathbb{D}_n) \longrightarrow G_k$, it turns out that B_n acts on the regular covering of $\mathbb{F}_k(\mathbb{D}_n)/S_k$, and the induced action on the Borel-Moore middle homology group of such a covering space is in fact the kth Bigelow-Krammer-Lawrence representation of B_n. In this way, for $k > 1$ we obtain faithful linear representations of B_n (see [10] for $k = 2$ and [36] for $k > 2$). We refer the reader to [30] for a complete description of these constructions. In what follows, we first motivate the choice of the above projections $p_k : B_k(\mathbb{D}_n) \longrightarrow G_k$ using the lower central series of the corresponding groups. We then explain how the study of the lower central series of surface braid groups can be used to obtain the representations given in [1]. We will define also the notion of extension of a representation in a completely algebraic manner, and its obstructions.

If we wish to consider surjections of $B_k(\mathbb{D}_n)$ onto some group G'_k, in order to obtain a linear representation using the approach described above, the group G'_k should be Abelian. Considering the short exact sequence (MB) on the level of Abelianisation, we obtain the following commutative diagram of short exact sequences:

$$\begin{tikzcd} 1 \arrow[r] & B_k(\mathbb{D}_n) \arrow[r] \arrow[d, "\bar{q}_k"] & B_{k,n} \arrow[r] \arrow[d, "r_{k,n}"] & B_n \arrow[r] \arrow[d, "r_n"] & 1 \\ 1 \arrow[r] & \ker \bar{\psi}_k \arrow[r] & B_{k,n}/\Gamma_2(B_{k,n}) \arrow[r] & B_n/\Gamma_2(B_n) \arrow[r] & 1 \end{tikzcd} \tag{2}$$

where $r_{k,n}$ and r_n are Abelianisation homomorphisms, and $\bar{\psi}_k$ is the homomorphism satisfying $\bar{\psi}_k \circ r_{k,n} = r_n \circ \psi_k$. It is straightforward to show that for all $k \geq 1$, $\ker \bar{\psi}_k$ and $\bar{q}_k$ coincide respectively with the group G_k and the morphism p_k considered in the Bigelow-Krammer-Lawrence representations.

3 Exact sequences for surface braid groups

We now return to the general case. Let Σ be an orientable surface. The map $\mathbb{F}_{k+n}(\Sigma)/(S_k \times S_n) \longrightarrow \mathbb{F}_n(\Sigma)/S_n$ given by forgetting the first k coordinates is a locally-trivial fibration whose fibre may be identified with $\mathbb{F}_k(\Sigma \setminus \{x_1, \ldots, x_n\})/S_k$. As in the case of pure braid groups [18], the long exact sequence in homotopy of this fibration yields a short exact sequence.

Lemma 3.1. *Let $k, n \in \mathbb{N}$. The Fadell-Neuwirth fibration $\mathbb{F}_{k+n}(\Sigma) \longrightarrow \mathbb{F}_n(\Sigma)$ induces the short exact sequence:*

$$1 \longrightarrow B_k(\Sigma \setminus \{x_1, \ldots, x_n\}) \longrightarrow B_{k,n}(\Sigma) \longrightarrow B_n(\Sigma) \longrightarrow 1\,, \tag{MSB}$$

where we suppose that $n \geq 3$ if $\Sigma = \mathbb{S}^2$.

In what follows we shall refer to the above short exact sequence as (MSB) (mixed surface braid groups sequence), and we denote its restriction to the corresponding pure braid groups

$$1 \longrightarrow P_k(\Sigma \setminus \{x_1, \ldots, x_n\}) \longrightarrow P_{k+n}(\Sigma) \longrightarrow P_n(\Sigma) \longrightarrow 1\,, \tag{SPB}$$

by (SPB) (surface pure braid groups sequence). If Σ is the disc $\mathbb{D}^2$, we recover the sequence considered in the previous section that gives rise to the Bigelow-Krammer Lawrence representation. In a similar manner, we may study (MSB) in order to find representations of $B_n(\Sigma)$.

We have the following commutative diagram involving the short exact sequences (1), (SPB) and (MSB):

$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & 1 \arrow[d] & \\
1 \arrow[r] & P_k(\Sigma \setminus \{x_1, \ldots, x_n\}) \arrow[r] \arrow[d] & P_{k+n}(\Sigma) \arrow[r] \arrow[d] & P_n(\Sigma) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & B_k(\Sigma \setminus \{x_1, \ldots, x_n\}) \arrow[r] \arrow[d] & B_{k,n}(\Sigma) \arrow[r] \arrow[d] & B_n(\Sigma) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & S_k \arrow[r] \arrow[d] & S_k \times S_n \arrow[r] \arrow[d] & S_n \arrow[r] \arrow[d] & 1 \\
 & 1 & 1 & 1 &
\end{tikzcd} \tag{3}$$

where the vertical arrows between (SPB) and (MSB) are inclusions, and the second vertical sequence is obtained by restricting the exact sequence (1) for $k+n$ strings to $B_{k,n}(\Sigma)$. The third row of symmetric groups splits as a direct product.

The following lemma summarises some of the known results for the splitting problem for the sequences (SPB) and (MSB) (we refer to [19, 25] for the case of $\mathbb{S}^2$.

Lemma 3.2. *Let Σ be a compact, connected orientable surface different from $\mathbb{S}^2$.*

(a) Suppose that Σ has empty boundary.

(i) If Σ is the 2-torus $\mathbb{T}^2$ then (SPB) splits for all $k, n \in \mathbb{N}$.

(ii) If $n = 1$ then both (SPB) and (MSB) split for all $k \in \mathbb{N}$.

(iii) Let $n \geq 2$ and $k \in \mathbb{N}$. If $\Sigma \neq \mathbb{T}^2$, then (SPB) does not split. If further $k = 1$ then (MSB) does not split.

(b) If Σ has non-empty boundary then (SPB) and (MSB) split for all $k, n \in \mathbb{N}$.

Proof. (a) (i) The statement is a consequence of [18] and the fact that $\mathbb{T}^2$ admits a non-vanishing vector field.

(ii) Suppose that $n = 1$. Then (SPB) splits using [22] in the orientable case. The fact that the upper right-hand vertical arrow $P_1(\Sigma) \longrightarrow B_1(\Sigma)$ of the diagram (3) is the identity yields a section for $B_{k,1}(\Sigma) \longrightarrow B_1(\Sigma)$.

(iii) For (SPB), this follows from [22]. Next, let $k = 1$, suppose that (MSB) splits, and let s be a section for $B_{1,n}(\Sigma) \longrightarrow B_n(\Sigma)$. For $m \in \mathbb{N}$, let $\tau_m : B_m(\Sigma) \longrightarrow S_m$ denote the usual permutation homomorphism that appears in the short exact sequence (1). If $x \in P_n(\Sigma)$ then $\tau_n(x) = 1$, and so $\tau_{1+n}(s(x)) = 1$ by commutativity of the diagram (12) and the fact that $k = 1$ (by abuse of notation, τ_{1+n} also denotes the restriction of τ_{1+n} to $B_{1,n}(\Sigma)$). Hence $s(x) \in P_{1+n}(\Sigma)$. Since $P_{1+n}(\Sigma) \longrightarrow P_n(\Sigma)$ is the restriction of $B_{1,n}(\Sigma) \longrightarrow B_n(\Sigma)$ to $P_{1+n}(\Sigma)$, the restriction of s to $P_n(\Sigma)$ gives rise to a section for (SPB), and so we obtain a contradiction.

(b) Suppose that Σ has non-empty boundary, and let C be a boundary component of Σ. Then $\Sigma' = \Sigma \setminus C$ is homeomorphic to a compact surface with a single point deleted. The fact that Σ and Σ' are homotopy equivalent implies that their configuration spaces are also homotopy equivalent, and hence the pure braid groups (resp. braid groups, mixed braid groups) of Σ are isomorphic to the corresponding pure braid groups (resp. braid groups, mixed braid groups) of Σ'. Applying the methods of [22], the short exact sequences (SPB) and (MSB) split for Σ', and so split for Σ. $\square$

Taking into account the above discussion concerning the existence of braid group representations via the induced action on first homology, this lemma indicates that one might use (MSB) to look for representations of surface braid groups in the case where the boundary is non empty.

4 Lower central series for surface braid groups

4.1 Definitions and known results

In this section we will give some results on the lower central series of (mixed) surface braid groups that will turn out useful in the next section when we come

to study their representations. Given a group G, we recall that the *lower central series* of G is the filtration $G = \Gamma_1(G) \supseteq \Gamma_2(G) \supseteq \cdots$, where $\Gamma_i(G) = [G, \Gamma_{i-1}(G)]$ for $i \geq 2$. The group G is said to be *perfect* if $G = \Gamma_2(G)$. Following P. Hall, for a group-theoretic property $\mathcal{P}$, a group G is said to be *residually* $\mathcal{P}$ if for any (non-trivial) element x in G, there exists a group H with the property $\mathcal{P}$ and a surjective homomorphism $\varphi : G \longrightarrow H$ such that $\varphi(x) \neq 1$. It is well known that a group G is residually nilpotent if and only if $\bigcap_{i\geq 1} \Gamma_i(G) = \{1\}$.

In what follows, we denote a compact, connected orientable surface with one boundary component of positive genus g by $\widehat{\Sigma}_g$. The surface $\widehat{\Sigma}_g \setminus \{x_1, \ldots, x_n\}$ will be denoted by $\widehat{\Sigma}_{g,n}$. We will focus on $\widehat{\Sigma}_g$ since, according to Lemma 3.2, for these particular surfaces the sequence (MSB) splits. The following result is well known (see [8, 29] for instance).

Proposition 4.1. *Let B_n be the Artin braid group on $n \geq 3$ strands. Then $\Gamma_1(B_n)/\Gamma_2(B_n) \cong \mathbb{Z}$ and $\Gamma_2(B_n) = \Gamma_3(B_n)$.*

The case of braid groups of orientable surfaces of genus at least one, is much richer (the cases of the sphere $\mathbb{S}^2$ and the punctured sphere were studied in [23]): related statements for $\widehat{\Sigma}_g$ may be summarised as follows (see [5, 8, 29] for similar results for the other orientable surfaces).

Theorem 4.2. *([8]) Let $g \geq 1$ and $n \geq 3$. Then:*

(a) $\Gamma_1(B_n(\widehat{\Sigma}_g))/\Gamma_2(B_n(\widehat{\Sigma}_g)) = \mathbb{Z}^{2g} \oplus \mathbb{Z}_2$.

(b) $\Gamma_2(B_n(\widehat{\Sigma}_g))/\Gamma_3(B_n(\widehat{\Sigma}_g)) = \mathbb{Z}$.

(c) $\Gamma_3(B_n(\widehat{\Sigma}_g)) = \Gamma_4(B_n(\widehat{\Sigma}_g))$. *Moreover* $\Gamma_3(B_n(\widehat{\Sigma}_g))$ *is perfect for* $n \geq 5$.

(d) $B_n(\widehat{\Sigma}_g)$ *is not residually nilpotent.*

4.2 Group presentations for surface braid groups and their 3-commutator groups

Theorem 4.3. *([8]) Let $n \geq 1$. The group $B_n(\widehat{\Sigma}_g)$ admits the following group presentation:*
Generators: $a_1, b_1, \ldots, a_g, b_g, \sigma_1, \ldots, \sigma_{n-1}$.
Relations:

$$\sigma_i\sigma_j = \sigma_j\sigma_i \text{ if } |i-j| \geq 2 \tag{4}$$
$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \text{ for all } 1 \leq i \leq n-2 \tag{5}$$
$$c_i\sigma_j = \sigma_j c_i \text{ for all } j \geq 2,\ c_i = a_i \text{ or } b_i \text{ and } i = 1, \ldots, g \tag{6}$$
$$c_i\sigma_1 c_i\sigma_1 = \sigma_1 c_i\sigma_1 c_i \text{ for } c_i = a_i \text{ or } b_i \text{ and } i = 1, \ldots, g \tag{7}$$
$$a_i\sigma_1 b_i = \sigma_1 b_i\sigma_1 a_i\sigma_1 \text{ for } i = 1, \ldots, g \tag{8}$$
$$c_i\sigma_1^{-1} c_j\sigma_1 = \sigma_1^{-1} c_j\sigma_1 c_i \text{ for } c_i = a_i \text{ or } b_i,\ c_j = a_j \text{ or } b_j \text{ and } 1 \leq j < i \leq g \tag{9}$$

In Figure 1 we recall a geometric interpretation of the generators of $B_n(\widehat{\Sigma}_g)$; we represent $\widehat{\Sigma}_g$ as a polygon with $4g$ sides, equipped with the standard identification of edges and one boundary component. We may consider braids as paths on the polygon, which we draw with the usual 'over and under' information at the crossing points. For the braid a_i (respectively b_j), the only non-trivial string is the first one, which passes through the wall α_i (respectively the wall β_j). The braids $\sigma_1, \ldots, \sigma_{n-1}$ are the standard braid generators of the disc. One can easily write a braid represented by a loop of the first strand around the boundary component as the composition of the generators (see for instance Section 2.2 of [6]).

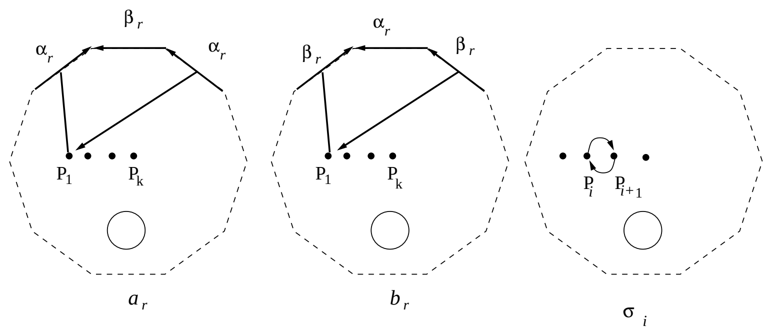

Figure 1: The generators $\sigma_1, \ldots, \sigma_{k-1}, a_1, b_1, \ldots, a_g, b_g$

From Theorem 4.3 one may deduce the following result.

Lemma 4.4. *([8]) Let $n \geq 3$. The quotient group $B_n(\widehat{\Sigma}_g)/\Gamma_3(B_n(\widehat{\Sigma}_g))$ admits the following group presentation:*
Generators:
$a_1, b_1, \ldots, a_g, b_g, \sigma$.
Relations:

$$a_1, b_1, \ldots, a_g, b_g \text{ and } \sigma \text{ commute pairwise except } (a_i, b_i)_{i=1,\ldots,g}; \tag{10}$$
$$[a_1, b_1] = \cdots = [a_g, b_g] = \sigma^2. \tag{11}$$

We point out that the corresponding result proved in [8, Theorem 1] is in fact for closed surfaces but the proof given there can be repeated verbatim to prove Lemma 4.4.

The following Corollary is a straightforward consequence of Lemma 4.4.

Corollary 4.5. *Let $n \geq 3$. The group $B_n(\widehat{\Sigma}_g)/\Gamma_3(B_n(\widehat{\Sigma}_g))$ is isomorphic to the semi-direct product*
$$G_g = (\mathbb{Z} \times \mathbb{Z}^g) \rtimes \mathbb{Z}^g.$$
More precisely, the first factor $\mathbb{Z}$ is central and is generated by σ, the second factor $\mathbb{Z}^g$ is generated by $\{a_1, \ldots, a_g\}$, and the third factor $\mathbb{Z}^g$ is generated by $\{b_1, \ldots, b_g\}$. Any generator b_j (for $1 \leq j \leq g$) acts trivially on $a_1, \ldots, a_{j-1}, a_{j+1}$

and $b_j a_j b_j^{-1} = \sigma^{-2} a_j$. *Hence,* $B_n(\widehat{\Sigma}_g)/\Gamma_3(B_n(\widehat{\Sigma}_g))$ *is a central extension of* $\mathbb{Z}^{2g}$ *by* $\mathbb{Z}$.

Thus every element of $B_n(\Sigma_g)/\Gamma_3(B_n(\Sigma_g))$ can be written in a unique way in the form $\sigma^p \prod_{i=1}^{g} a_i^{m_i} \prod_{i=1}^{g} b_i^{n_i}$.

4.3 Group presentations for mixed braid groups of surfaces and their 3-commutator groups

Following the standard definition of a Coxeter system, we introduce the notion of a *surface braid group system*.

Definition 4.6. *Let G be a group. Let $S = \{\sigma_1, \cdots, \sigma_{k-1}\}$, $AB = \{a_1, \cdots, a_g, b_1, \cdots, b_g\}$ and $Z = \{\zeta_1, \ldots, \zeta_n\}$ be subsets of G such that $k \geq 1$ and $g, n \geq 0$. We say that (G, S, AB, Z) is a* surface braid group system *if G admits the following group presentation:*

$$\begin{array}{ll}
\sigma_i\sigma_j = \sigma_j\sigma_i; & |i-j| \geq 2\\
\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}; & 1 \leq i \leq k-2\\
a_i\sigma_j = \sigma_j a_i \text{ and } b_i\sigma_j = \sigma_j b_i; & j \in \{2, \ldots, k-1\},\ i \in \{1, \ldots, g\}\\
a_i\sigma_1 a_i\sigma_1 = \sigma_1 a_i\sigma_1 a_i \text{ and } b_i\sigma_1 b_i\sigma_1 = \sigma_1 b_i\sigma_1 b_i & i \in \{1, \ldots, g\}\\
a_i\sigma_1 b_i = \sigma_1 b_i\sigma_1 a_i\sigma_1; & i \in \{1, \ldots, g\}\\
c_i(\sigma_1^{-1}c_j\sigma_1) = (\sigma_1^{-1}c_j\sigma_1)c_i; & c_i \in \{a_i, b_i\}, c_j \in \{a_j, b_j\},\ j < i\\
\zeta_j\sigma_i = \sigma_i\zeta_j & i \neq 1;\\
(\sigma_1^{-1}\zeta_j\sigma_1)a_\ell = a_\ell(\sigma_1^{-1}\zeta_j\sigma_1) & \\
(\sigma_1^{-1}\zeta_j\sigma_1)b_\ell = b_\ell(\sigma_1^{-1}\zeta_j\sigma_1); & \\
(\sigma_1^{-1}\zeta_j\sigma_1)\zeta_\ell = \zeta_\ell(\sigma_1^{-1}\zeta_j\sigma_1) & j < \ell;\\
(\sigma_1\zeta_j\sigma_1)\zeta_j = \zeta_j(\sigma_1\zeta_j\sigma_1). &
\end{array}$$

The following Lemma is a straightforward consequence of the group presentations given in [6].

Lemma 4.7. *(i) There exist $S = \{\sigma_1, \cdots, \sigma_n\}$ and $AB = \{a_1, \cdots, a_g, b_1, \cdots, b_g\}$ such that $(B_n(\widehat{\Sigma}_g), S, AB, \emptyset)$ is a surface braid group system.*

(ii) There exist $S = \{\sigma_1, \cdots, \sigma_k\}$, $AB = \{a_1, \cdots, a_g, b_1, \cdots, b_g\}$ and $Z = \{\zeta_1, \ldots, \zeta_n\}$ such that $(B_k(\widehat{\Sigma}_{g,n}), S, AB, Z)$ is a surface braid group system.

Proposition 4.8. *The group $B_{k,n}(\widehat{\Sigma}_g)$ admits the following group presentation:*
Generating set: *$S \cup \tilde{S} \cup AB \cup \widetilde{AB} \cup Z$ with*

$$\begin{array}{lllllll}
\tilde{S} & = & \{\tilde{\sigma}_1, \ldots, \tilde{\sigma}_{n-1}\}, & S & = & \{\sigma_1, \ldots, \sigma_{k-1}\},\\
\widetilde{AB} & = & \{\tilde{a}_1, \tilde{b}_1, \ldots, \tilde{a}_g, \tilde{b}_g\}, & AB & = & \{a_1, b_1, \ldots, a_g, b_g\},\\
Z & = & \{\zeta_1, \cdots, \zeta_n\}. & & &
\end{array}$$

Relations:

(a) the relations associated with the system $(B_k(\widehat{\Sigma}_{g,n}), S, AB, Z)$;

(b) the relations associated with the system $(B_n(\widehat{\Sigma}_g), \tilde{S}, \widetilde{AB}, \emptyset)$;

(c) the relations describing the action of $B_n(\widehat{\Sigma}_g)$ *on* $B_k(\widehat{\Sigma}_{g,n})$;

(i) $\tilde{\sigma}_i\sigma_j\tilde{\sigma}_i^{-1} = \tilde{a}_i\sigma_j\tilde{a}_i^{-1} = \tilde{b}_i\sigma_j\tilde{b}_i^{-1} = \sigma_j$;

(ii) $\tilde{\sigma}_i a_j\tilde{\sigma}_i^{-1} = a_j$; $\tilde{\sigma}_i b_j\tilde{\sigma}_i^{-1} = b_j$

$$(iii)\ \begin{cases} \tilde{\sigma}_i\zeta_{i+1}\tilde{\sigma}_i^{-1} = \zeta_i; & \\ \tilde{\sigma}_i\zeta_i\tilde{\sigma}_i^{-1} = \zeta_i^{-1}\zeta_{i+1}\zeta_i; & \\ \tilde{\sigma}_i\zeta_j\tilde{\sigma}_i^{-1} = \zeta_j, & j \neq i, i+1; \end{cases} \quad \begin{cases} \tilde{a}_i\zeta_1\tilde{a}_i^{-1} = \zeta_1^{a_i\zeta_1}; & \\ \tilde{b}_i\zeta_1\tilde{b}_i^{-1} = \zeta_1^{b_i\zeta_1}; & \\ \tilde{a}_i\zeta_j\tilde{a}_i^{-1} = \zeta_j^{[a_i^{-1},\zeta_1^{-1}]} & j \neq 1; \\ \tilde{b}_i\zeta_j\tilde{b}_i^{-1} = \zeta_j^{[b_i^{-1},\zeta_1^{-1}]} & j \neq 1; \end{cases}$$

$$(iv)\ \begin{cases} \tilde{a}_i a_i\tilde{a}_i^{-1} = \zeta_1^{-1}a_i\zeta_1; & \\ \tilde{a}_i a_j\tilde{a}_i^{-1} = a_j^{[a_i^{-1},\zeta_1^{-1}]} & i > j; \\ \tilde{a}_i a_\ell\tilde{a}_i^{-1} = a_\ell, & \ell > i; \end{cases} \quad \begin{cases} \tilde{b}_i b_i\tilde{b}_i^{-1} = \zeta_1^{-1}b_i\zeta_1; & \\ \tilde{b}_i b_j\tilde{b}_i^{-1} = b_j^{[b_i^{-1},\zeta_1^{-1}]} & i > j; \\ \tilde{b}_i b_\ell\tilde{b}_i^{-1} = b_\ell, & \ell > i; \end{cases}$$

$$(v)\ \begin{cases} \tilde{a}_i b_i\tilde{a}_i^{-1} = b_i\zeta_1; & \\ \tilde{a}_i b_j\tilde{a}_i^{-1} = b_j^{[a_i^{-1},\zeta_1^{-1}]} & i > j; \\ \tilde{a}_i b_\ell\tilde{a}_i^{-1} = b_\ell, & \ell > i \end{cases} \quad \begin{cases} \tilde{b}_i a_i\tilde{b}_i^{-1} = \zeta_1^{-1}a_i[b_i^{-1},\zeta_1^{-1}]; & \\ \tilde{b}_i a_j\tilde{b}_i^{-1} = a_j^{[b_i^{-1},\zeta_1^{-1}]} & i > j; \\ \tilde{b}_i a_\ell\tilde{b}_i^{-1} = a_\ell, & \ell > i; \end{cases}$$

where $a^b := b^{-1}ab$.

Proof. As we recalled in Section 2, the short exact sequence (MSB): $1 \longrightarrow B_k(\widehat{\Sigma}_{g,n}) \longrightarrow B_{k,n}(\widehat{\Sigma}_g) \longrightarrow B_n(\widehat{\Sigma}_g) \longrightarrow 1$ splits. Therefore, the group $B_{k,n}(\widehat{\Sigma}_g)$ is isomorphic to $B_n(\widehat{\Sigma}_g) \ltimes B_k(\widehat{\Sigma}_{g,n})$. We may interpret the braids depicted in Figure 2 as geometric representatives of generators of $B_k(\widehat{\Sigma}_{g,n})$, and those depicted in Figure 3 as the coset representatives of generators of $B_n(\widehat{\Sigma}_g)$ in $B_n(\widehat{\Sigma}_g)$. The result follows by a straightforward verification of the corresponding geometric braids, see for instance Figure 4. □

As in the case of $B_n(\widehat{\Sigma}_g)/\Gamma_3(B_n(\widehat{\Sigma}_g))$ (Corollary 4.5), we may obtain a group presentation of $B_{k,n}(\widehat{\Sigma}_g)\,/\Gamma_3(B_{k,n}(\widehat{\Sigma}_g))$ from the previous proposition, and decompose the group as a semi-direct product.

Lemma 4.9. *Let* $k, n \geq 3$. *Then the group* $B_{k,n}(\widehat{\Sigma}_g)\,/\Gamma_3(B_{k,n}(\widehat{\Sigma}_g))$ *has the following group presentation:*
Generators: $\sigma, \tilde{\sigma}, \zeta, a_1, b_1, \ldots, a_g, b_g, \tilde{a}_1, \tilde{b}_1, \ldots, \tilde{a}_g, \tilde{b}_g$.
Relations:

(a) $[\sigma, a_i] = [\sigma, b_i] = [\tilde{\sigma}, \tilde{a}_i] = [\tilde{\sigma}, \tilde{b}_i] = [\tilde{\sigma}, a_i] = [\tilde{\sigma}, b_i] = [\sigma, \tilde{a}_i] = [\sigma, \tilde{b}_i] = [\sigma, \tilde{\sigma}] = 1$;

(b) $[a_i, a_j] = [a_i, b_j] = 1$ *for* $i \neq j$ *and* $[a_j, b_j] = \sigma^2$;

(c) $[\tilde{a}_i, \tilde{a}_j] = [\tilde{a}_i, \tilde{b}_j] = 1$ *for* $i \neq j$ *and* $[\tilde{a}_j, \tilde{b}_j] = \tilde{\sigma}^2$

(d) $[a_i, \tilde{a}_j] = [b_i, \tilde{b}_j] = 1$ *for any pair* (i, j);

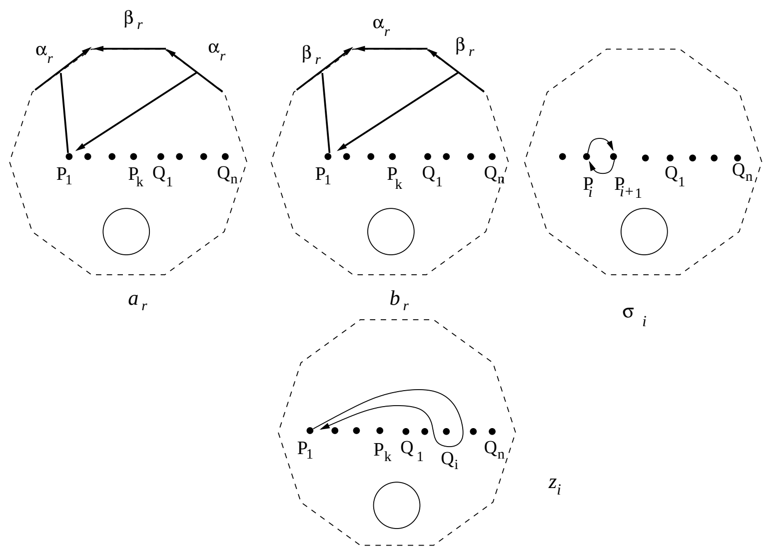

Figure 2: The generators $\sigma_1, \ldots, \sigma_{k-1}, a_1, b_1, \ldots, a_g, b_g, \zeta_1, \cdots, \zeta_n$

(e) $[b_i, \tilde{a}_j] = [\tilde{b}_j, a_i] = 1$ *for* $i \neq j$ *and* $[b_i, \tilde{a}_i] = [\tilde{b}_i, a_i] = \zeta$;

(f) $[\zeta, a_i] = [\zeta, b_i] = [\zeta, \tilde{a}_i] = [\zeta, \tilde{b}_i] = [\zeta, \sigma] = [\zeta, \tilde{\sigma}] = 1$.

Proof. Consider the group presentation of $B_{k,n}(\widehat{\Sigma}_g)$ given in Proposition 4.8 and the map $p : B_{k,n}(\widehat{\Sigma}_g) \longrightarrow B_{k,n}(\widehat{\Sigma}_g) / \Gamma_3(B_{k,n}(\widehat{\Sigma}_g))$. It follows from the braid relations $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ and $\tilde{\sigma}_i\tilde{\sigma}_{i+1}\tilde{\sigma}_i = \tilde{\sigma}_{i+1}\tilde{\sigma}_i\tilde{\sigma}_{i+1}$ that $\sigma_i = \sigma_{i+1}[\sigma_i, \sigma_{i+1}]$ and $\tilde{\sigma}_i = \tilde{\sigma}_{i+1}[\tilde{\sigma}_i, \tilde{\sigma}_{i+1}]$. But $[\tilde{\sigma}_i, \tilde{\sigma}_{i+1}]$ and $[\sigma_i, \sigma_{i+1}]$ belong to $\Gamma_2(B_{k,n}(\widehat{\Sigma}_g))$. Their images under p therefore belong to the centre of $B_{k,n}(\widehat{\Sigma}_g) / \Gamma_3(B_{k,n}(\widehat{\Sigma}_g))$. Since $p(\sigma_i)$ and $p(\sigma_{i+1})$ are center elements, the equality $p(\sigma_i\sigma_{i+1}\sigma_i) = p(\sigma_{i+1}\sigma_i\sigma_{i+1})$ implies that $p(\sigma_i) = p(\sigma_{i+1})$. We denote the image of σ_i under p by σ. Similarly, the $\tilde{\sigma}_j$ have the same image under p, which we denote by $\tilde{\sigma}$ in what follows. By abuse of notation, we also denote $p(a_i), p(b_i), p(\tilde{a}_i)$ and $p(\tilde{b}_i)$ by $a_i, b_i, \tilde{a}_i$ and $\tilde{b}_i$ respectively. Using the fact that $\sigma_i a_j = a_j \sigma_i$ for $i \neq j$, we obtain $\sigma a_j = a_j \sigma$ for all j. Similarly, we have $\sigma b_j = b_j \sigma$, $\sigma \tilde{a}_j = \tilde{a}_j \sigma$, $\tilde{\sigma}\tilde{b}_j = \tilde{b}_j\tilde{\sigma}$ and $\tilde{\sigma} p(\zeta_j) = p(\zeta_j)\tilde{\sigma}$. We deduce from relations $a_i(\sigma_1^{-1} a_j \sigma_1) = (\sigma_1^{-1} a_j \sigma_1) a_i$ for $i \neq j$ that $a_i a_j = a_j a_i$. Similarly we have $a_i b_j = b_j a_i$, $\tilde{a}_i\tilde{a}_j = \tilde{a}_j\tilde{a}_i$ and $\tilde{b}_i\tilde{a}_j = \tilde{a}_j\tilde{b}_i$ for $i \neq j$. For the same reason, we have $\tilde{a}_i p(\zeta_j) = p(\zeta_j)\tilde{a}_i$ and $\tilde{b}_i p(\zeta_j) = p(\zeta_j)\tilde{b}_i$. Now, from the equality $a_j \sigma b_j = \sigma b_j \sigma a_j \sigma$, we deduce the relation $[a_j, b_j] = \sigma^2$. Similarly we have $[\tilde{a}_j, \tilde{b}_j] = \tilde{\sigma}^2$. From Relations (i) and (ii) of the presentation of $B_{k,n}(\widehat{\Sigma}_g)$, we have $[\tilde{\sigma}, a_i] = [\tilde{\sigma}, b_i] = [\sigma, \tilde{a}_i] = [\sigma, \tilde{b}_i] = [\sigma, \tilde{\sigma}] = 1$. From Relations (iii) we see that $\sigma p(\zeta_{i+1})\sigma^{-1} = \zeta_i = \sigma_j \zeta_i \sigma_j^{-1}$ with $j \neq i, i+1$. Then, all the ζ_i have the same image under p, which we denote by ζ. Then $[\zeta, \tilde{a}_i] = [\zeta, \tilde{b}_i] = [\zeta, \sigma] = [\zeta, \tilde{\sigma}] = 1$. Still using Relations (iii), we have $[\zeta, a_i] = [\zeta, b_i] = 1$. From Relations (iv) we

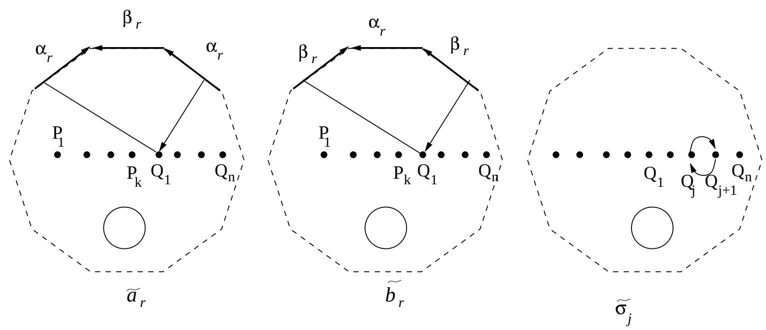

Figure 3: The generators $\tilde{\sigma}_1, \ldots, \tilde{\sigma}_{n-1}, \tilde{a}_1, \tilde{b}_1, \ldots, \tilde{a}_g, \tilde{b}_g$

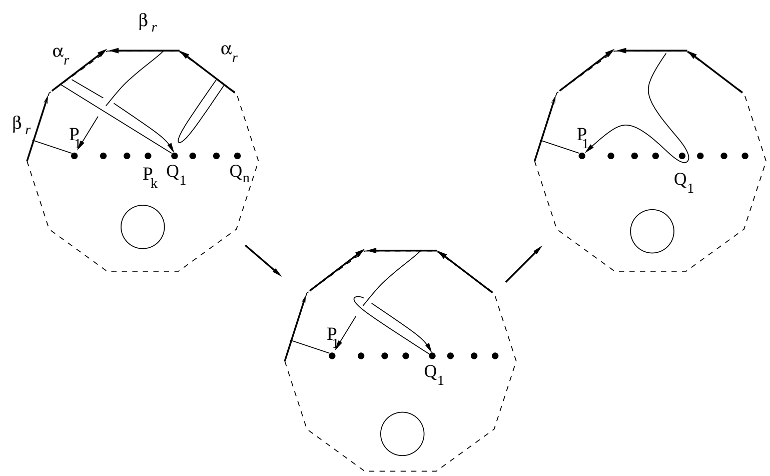

Figure 4: The braids $\tilde{a}_i b_i \tilde{a}_i^{-1}$ and $b_i \zeta_1$ are isotopic.

obtain $[a_i, \tilde{a}_j] = [b_i, \tilde{b}_j] = 1$, and from Relations (v) we get $[b_i, \tilde{a}_j] = [\tilde{b}_j, a_i] = 1$ for $i \neq j$ and $[b_i, \tilde{a}_i] = [\tilde{b}_i, a_i] = \zeta$. Then, the defining relations of the presentation given in the lemma hold in $B_{k,n}(\widehat{\Sigma}_g) / \Gamma_3(B_{k,n}(\widehat{\Sigma}_g))$ if one takes $\sigma = p(\sigma_1), \tilde{\sigma} = p(\tilde{\sigma}_1), \zeta = p(\zeta_1), a_i = p(a_i), b_i = p(b_i), \tilde{a}_i = p(\tilde{a}_i)$ and $\tilde{b}_i = p(\tilde{b}_i)$. Conversely, let G be the group defined by the presentation given in the lemma. We may check without difficulty that we have a group homomorphism from $B_{k,n}(\widehat{\Sigma}_g)$ to G that sends $\sigma_i, \tilde{\sigma}_j, a_{i'}, b_{i''}, \tilde{a}_{j'}, \tilde{a}_{j''}$ and ζ_ℓ to $\sigma, \tilde{\sigma}, a_{i'}, b_{i''}, \tilde{a}_{j'}, \tilde{a}_{j''}$ and ζ respectively. In order to prove that G is isomorphic to $B_{k,n}(\widehat{\Sigma}_g) / \Gamma_3(B_{k,n}(\widehat{\Sigma}_g))$, we need to prove that in G, the equality $[a, [b, c]] = 1$ holds for all a, b, c in G. Using the equality $[a, bc] = [a, b]b[a, c]b^{-1}$, it is enough to consider the case where a, b and c are defining generators of G. But in these particular cases, the equality clearly holds since $\sigma, \tilde{\sigma}$ and ζ belong to the centre of G, and therefore any commutator $[x, y]$, where x, y are defining generators, is in the centre of G. Thus G and

$B_{k,n}(\widehat{\Sigma}_g)\,/\Gamma_3(B_{k,n}(\widehat{\Sigma}_g))$ are isomorphic. □

Corollary 4.10. *Let $k, n \geq 3$. The group* $\dfrac{B_{k,n}(\widehat{\Sigma}_g)}{\Gamma_3(B_{k,n}(\widehat{\Sigma}_g))}$ *is isomorphic to the semi-direct product:*

$$H_g := \left(\mathbb{Z}^3 \times \mathbb{Z}^{2g}\right) \rtimes \mathbb{Z}^{2g}.$$

More precisely, the first factor $\mathbb{Z}^3$ is central and is generated by $\sigma, \tilde{\sigma}, \zeta$, the second factor $\mathbb{Z}^{2g}$ is generated by $\{a_1, \ldots, a_g, \tilde{a}_1 \ldots, \tilde{a}_g\}$, and the third factor $\mathbb{Z}^{2g}$ is generated by $\{b_1, \ldots, b_g, \tilde{b}_1 \ldots, \tilde{b}_g\}$.

The actions defining the above semi-direct product are as described in Lemma 4.9. Then $B_{k,n}(\widehat{\Sigma}_g)\,/\Gamma_3(B_{k,n}(\widehat{\Sigma}_g))$ can be seen as a central extension of $\mathbb{Z}^{4g}$ by $\mathbb{Z}^3$. We remark that the action of any generator on the previous ones is trivial except in at most two cases: for instance the only non-trivial actions of $\tilde{a}_j$ are $\tilde{a}_j \tilde{b}_j \tilde{a}_j^{-1} = \tilde{\sigma}^2 \tilde{b}_j$ and $\tilde{a}_j b_j \tilde{a}_j^{-1} = \zeta^{-2} b_j$. Finally, notice that every element w of $B_{k,n}(\widehat{\Sigma}_g)\,/\Gamma_3(B_{k,n}(\widehat{\Sigma}_g))$ may be written in a unique way in the form $w = \sigma^p \tilde{\sigma}^q \zeta^r \prod_{i=1}^{g} a_i^{m_i} \tilde{a}_i^{M_i} \prod_{i=1}^{g} b_i^{n_i} \tilde{b}_i^{N_i}$.

5 Mixed surface braid sequences and representations of surface braid groups

As in previous sections, we will focus on connected orientable surfaces of positive genus with a single boundary component. One might ask whether it is possible to use the short exact sequence (MSB) to obtain representations of $B_n(\widehat{\Sigma}_g)$ which are extensions of Bigelow-Krammer-Lawrence representations of B_n. First, let us denote by $\iota_n : B_n \longrightarrow B_n(\widehat{\Sigma}_g)$ and $\iota_{k,n} : B_k(\mathbb{D}_n) \longrightarrow B_k(\widehat{\Sigma}_{g,n})$ the homomorphisms induced by the inclusion of $\mathbb{D}^2$ into $\widehat{\Sigma}_g$ and of $\mathbb{D}_n$ into $\widehat{\Sigma}_{g,n}$ respectively. We recall that they are injective (see for instance [34]).

Remark 5.1. In particular $B_k(\mathbb{D}_n)$ is generated as a subgroup of $B_k(\widehat{\Sigma}_{g,n})$ by $\sigma_1, \ldots, \sigma_{k-1}$ and $\zeta_1, \ldots, \zeta_n$ (see for instance [4, 6]).

Given $\beta \in B_n$, we denote the induced action by conjugation of β on $B_k(\mathbb{D}_n)$, and abusing notation, the action of β (regarded as an element of $B_n(\widehat{\Sigma}_g)$) on $B_k(\widehat{\Sigma}_{g,n})$, by β_*. As in Section 2, for the case $k = 1$ we consider the length function $p_1 : B_1(\mathbb{D}_n) \longrightarrow G_1 = \mathbb{Z} = \langle t \rangle$, while for $k > 1$ we set G_k to be the group $\mathbb{Z}^2 = \langle q, t \rangle$. The corresponding homomorphism $p_k : B_k(\mathbb{D}_n) \longrightarrow G_k$ for $k > 1$ sends $\sigma_1, \ldots, \sigma_k$ to q and $\zeta_1, \ldots, \zeta_n$ to t. As we mentioned in Section 2, the fact that the action of B_n on $B_k(\mathbb{D}_n)$ commutes with $p_k : B_k(\mathbb{D}_n) \longrightarrow G_k$ implies that B_n acts on a regular covering. For $k = 1$ the induced action on homology gives rise to the Burau representation, while for $k > 1$ we obtain faithful linear representations of B_n.

Definition 5.2. *Let $P_k : B_k(\widehat{\Sigma}_{g,n}) \longrightarrow G_k(\widehat{\Sigma}_g)$ be a surjective homomorphism. We will say that the homomorphism P_k is a lifting extension of the map $p_k : B_k(\mathbb{D}_n) \longrightarrow G_k$ defined above if there exists an injective homomorphism $\bar{\iota}_k : G_k \longrightarrow G_k(\widehat{\Sigma}_g)$ such that $P_k \circ \iota_{k,n} = \bar{\iota}_k \circ p_k$, and if further for all $\beta \in B_n(\widehat{\Sigma}_g)$ there exists a homomorphism $\bar{\beta}_* : G_k(\widehat{\Sigma}_g) \longrightarrow G_k(\widehat{\Sigma}_g)$ such that $\bar{\beta}_* \circ P_k = P_k \circ \beta_*$.*

The second condition means that there is an induced action of $B_n(\widehat{\Sigma}_g)$ on the homology of the covering space of $\mathbb{F}_k(\widehat{\Sigma}_{g,n})/S_k$. In other words, the surjection P_k is a lifting extension of p_k if the following diagram commutes for any $\beta \in B_n$:

$$\begin{tikzcd}
G_k \arrow[rrr, hook, "\bar{\iota}_k"] \arrow[ddd, equal, "Id"'] & & & G_k(\widehat{\Sigma}_g) \arrow[ddd, "\bar{\beta}_*"] \\
& B_k(\mathbb{D}_n) \arrow[r, hook] \arrow[ul, "p_k"'] \arrow[d, "\beta_*"] & B_k(\widehat{\Sigma}_{g,n}) \arrow[ur, "P_k"] \arrow[d, "\beta_*"] & \\
& B_k(\mathbb{D}_n) \arrow[r, hook] \arrow[dl, "p_k"] & B_k(\widehat{\Sigma}_{g,n}) \arrow[dr, "P_k"] & \\
G_k \arrow[rrr, hook, "\bar{\iota}_k"] & & & G_k(\widehat{\Sigma}_g)
\end{tikzcd} \tag{12}$$

We remark that the middle homology group $H_k^{BM}\left(\widetilde{\dfrac{\mathbb{F}_k(\widehat{\Sigma}_{g,n})}{S_k}}\right)$ of the covering space $\widetilde{\dfrac{\mathbb{F}_k(\widehat{\Sigma}_{g,n})}{S_k}}$ of $\dfrac{\mathbb{F}_k(\widehat{\Sigma}_{g,n})}{S_k}$ is a free $\mathbb{Z}[G_k(\widehat{\Sigma}_g)]$-module (see Lemma 3.3 of [1]) and that some $\beta \in B_n(\widehat{\Sigma}_g)$ acts as a $\mathbb{Z}[G_k(\widehat{\Sigma}_g)]$-module morphism of $H_k^{BM}(\widetilde{\mathbb{F}_k(\widehat{\Sigma}_{g,n})/S_k})$ if and only if the map $\bar{\beta}_* : G_k(\widehat{\Sigma}_g) \longrightarrow G_k(\widehat{\Sigma}_g)$ defined above is the identity homomorphism (see Section 2 of [1]). If this property holds for all $\beta \in B_n(\widehat{\Sigma}_g)$ we thus obtain a representation of $B_n(\widehat{\Sigma}_g)$ in $Aut_{\mathbb{Z}[G_k(\widehat{\Sigma}_g)]}(H_k^{BM}\left(\widetilde{\dfrac{\mathbb{F}_k(\widehat{\Sigma}_{g,n})}{S_k}}\right))$ which is a linear representation if $G_k(\widehat{\Sigma}_g)$ is Abelian. The above discussion gives rise naturally to the following definition, which provides a notion of extension of the Bigelow-Krammer-Lawrence representations from B_n to $B_n(\widehat{\Sigma}_g)$.

Definition 5.3. *Let $P_k : B_k(\widehat{\Sigma}_{g,n}) \longrightarrow G_k(\widehat{\Sigma}_g)$ be a lifting extension of the homomorphism $p_k : B_k(\mathbb{D}_n) \longrightarrow G_k$. The homomorphism P_k is said to be a linear extension of p_k if for any $\beta \in B_n(\widehat{\Sigma}_g)$, we have that $\bar{\beta}_* = Id$ and that $G_k(\widehat{\Sigma}_g)$ is Abelian.*

The following proposition states that it is not in fact possible to extend the Bigelow-Krammer-Lawrence representations from B_n to $B_n(\widehat{\Sigma}_g)$.

Proposition 5.4. *There is no homomorphism $P_k : B_k(\widehat{\Sigma}_{g,n}) \longrightarrow G_k(\widehat{\Sigma}_g)$ that is a linear extension of $p_k : B_k(\mathbb{D}_n) \longrightarrow G_k$.*

Proof. This result is a reformulation of Lemma 2.6 of [1]. We sketch the proof. The action by conjugation of $B_n(\widehat{\Sigma}_g)$ on $B_k(\widehat{\Sigma}_{g,n})$ is described in Proposition 4.8, where $B_k(\widehat{\Sigma}_{g,n})$ is the subgroup of $B_{k,n}(\widehat{\Sigma}_g)$ generated by S, AB and Z. For any generator g, let us denote its image in $G_k(\widehat{\Sigma}_g)$ by $[g]$, and we consider $\tilde{b}_j$ as a generator of $B_n(\widehat{\Sigma}_g)$. Since P_k is a lifting extension of p_k, the action of $\tilde{b}_j$ by conjugation induces the homomorphism $\overline{(\tilde{b}_j)_*} : G_k(\widehat{\Sigma}_g) \longrightarrow G_k(\widehat{\Sigma}_g)$, which sends the element $[a_i]$ to $[a_i][\zeta_1]$. Since P_k is also a linear extension, $\overline{(\tilde{b}_j)_*}$ must coincide with the identity. One deduces that $[\zeta_1] = 1$, but this cannot be true since the hypothesis that P_k is a lifting extension of p_k implies that $[\zeta_1] = q$. $\square$

In what follows we show that for $k \geq 3$, there is a unique group $G_k(\widehat{\Sigma}_g)$ such that one may define a lifting extension $P_k : B_k(\widehat{\Sigma}_{g,n}) \longrightarrow G_k(\widehat{\Sigma}_g)$ of p_k. We first characterise the group $G_k(\widehat{\Sigma}_g)$ which was defined in Section 3.1 of [1] (with the notation G_Σ) in terms of lower central series. We then show that this group is the unique group admitting a lifting extension for p_k, and we construct the corresponding homomorphism $P_k : B_k(\widehat{\Sigma}_{g,n}) \longrightarrow G_k(\widehat{\Sigma}_g)$.

Consider the following diagram:

$$\begin{tikzcd}
1 \arrow[r] & B_k(\widehat{\Sigma}_{g,n}) \arrow[r] \arrow[d, "\bar{P}_k"] & B_{k,n}(\widehat{\Sigma}_g) \arrow[r] \arrow[d, "r_{k,n}"] & B_n(\widehat{\Sigma}_g) \arrow[r] \arrow[d, "r_n"] & 1 \\
1 \arrow[r] & \ker \bar{\psi}_k \arrow[r] & B_{k,n}(\widehat{\Sigma}_g)/\Gamma_3(B_{k,n}(\widehat{\Sigma}_g)) \arrow[r, "\bar{\psi}_k"] & B_n(\widehat{\Sigma}_g)/\Gamma_3(B_n(\widehat{\Sigma}_g)) \arrow[r] & 1
\end{tikzcd} \tag{13}$$

where, by abuse of notation, $r_{k,n}$ and r_n denote the canonical projections and $\bar{\psi}_k$ is the map such that $\bar{\psi}_k \circ r_{k,n} = r_n \circ \psi_k$. Following Lemma 4.9, we denote the generators of $B_{k,n}(\widehat{\Sigma}_g)/\Gamma_3(B_{k,n}(\widehat{\Sigma}_g))$ by $\sigma, \tilde{\sigma}, \zeta, a_1, b_1, \ldots, a_g, b_g, \tilde{a}_1, \tilde{b}_1, \ldots, \tilde{a}_g, \tilde{b}_g$.

Proposition 5.5. *The group $\ker \bar{\psi}_k$ admits the following presentation:*

Generating set: $\sigma, \zeta, a_1, b_1, \ldots, a_g, b_g$;
Relations:

(a) $[\sigma, a_i] = [\sigma, b_i] = 1$;

(b) $[a_i, a_j] = [a_i, b_j] = 1$ *for* $i \neq j$ *and* $[a_j, b_j] = \sigma^2$;

(c) $[\zeta, a_i] = [\zeta, b_i] = [\zeta, \sigma] = 1$.

Proof. Let us denote the subgroup of $B_{k,n}(\widehat{\Sigma}_g)/\Gamma_3(B_{k,n}(\widehat{\Sigma}_g))$ generated by $\sigma, \zeta, a_1, b_1, \ldots, a_g, b_g$ by K_g. From diagram (13), one deduces that the group K_g belongs to $\ker \bar{\psi}_k$. By Lemma 4.9, the group K_g is normal in $B_{k,n}(\widehat{\Sigma}_g)/\Gamma_3(B_{k,n}(\widehat{\Sigma}_g))$. A straightforward calculation shows that the quotient $B_{k,n}(\widehat{\Sigma}_g)/\Gamma_3(B_{k,n}(\widehat{\Sigma}_g))$ by

K_g is isomorphic to $B_n(\widehat{\Sigma}_g)/\Gamma_3(B_n(\widehat{\Sigma}_g))$, and therefore K_g coincides with $\ker \bar{\psi}_k$. To prove the proposition, it is therefore sufficient to check that the given set of relations is a complete set of relations for K_g. Let D_g be the (abstract) group with the group presentation given in the statement. Let $j : D_g \longrightarrow K_g$ be the (surjective) homomorphism sending every generator of D_g to the corresponding generator of K_g, and let $\iota : K_g \longrightarrow B_{k,n}(\widehat{\Sigma}_g)/\Gamma_3(B_{k,n}(\widehat{\Sigma}_g))$ be the (injective) homomorphism given in diagram (13). We claim that the composition $k = \iota \circ j$ is injective, and therefore D_g coincides with K_g. In fact, it follows from the group presentation of D_g that any element $w \in D_g$ can be written (in a unique way) as $w = \sigma^p \zeta^q \prod_{j=1}^{g} a_j^{m_j} \prod_{j=1}^{g} b_j^{n_j}$. We call this decomposition of w its normal form in D_g. On the other hand, any element $w' \in B_{k,n}(\widehat{\Sigma}_g)/\Gamma_3(B_{k,n}(\widehat{\Sigma}_g))$ may be written uniquely as $w' = \sigma^p \zeta^q \tilde{\sigma}^r \prod_{j=1}^{g}(a_j^{m_j} \tilde{a}_j^{M_j}) \prod_{j=1}^{g}(b_j^{n_j} \tilde{b}_j^{N_j})$. We call this decomposition of w' its normal form in $B_{k,n}(\widehat{\Sigma}_g)/\Gamma_3(B_{k,n}(\widehat{\Sigma}_g))$. Let $w \in D_g$ be written in its normal form. From the definition of $k : D_g \longrightarrow B_{k,n}(\widehat{\Sigma}_g)/\Gamma_3(B_{k,n}(\widehat{\Sigma}_g))$, $k(w)$ coincides with its normal form in $B_{k,n}(\widehat{\Sigma}_g)/\Gamma_3(B_{k,n}(\widehat{\Sigma}_g))$. Therefore $k(w) = 1$ implies that $w = 1$. □

Remark 5.6. Let $k \geq 3$. Then the group $\ker \bar{\psi}_k$ is the group G_Σ introduced in [1, Section 3]. The homomorphism $P_k : B_k(\widehat{\Sigma}_{g,n}) \longrightarrow \ker \bar{\psi}_k$ is the homomorphism $\Phi_\Sigma : B_k(\widehat{\Sigma}_{g,n}) \longrightarrow G_\Sigma$ also defined in [1, Section 3].

The homomorphism $\Phi_\Sigma : B_k(\widehat{\Sigma}_{g,n}) \longrightarrow G_\Sigma$ and the group G_Σ defined in Section 3 of [1] were constructed in a technical manner in order to obtain (non-linear) representations of surface braid groups subject to certain homological constraints (Definition 2.2 of [1]). Proposition 5.5 and Remark 5.6 show that such technical constructions coincide with objects arising in the lower central series. On the other hand, reinterpreting Lemma 3.1 and Theorem 4.3 of [1], we conclude that the unique lifting extension of $p_k : B_k(\mathbb{D}_n)$ to $B_k(\widehat{\Sigma}_{g,n})$ arises from the lower central series and the homomorphism $P_k : B_k(\widehat{\Sigma}_{g,n}) \longrightarrow \ker \bar{\psi}_k$, as we shall now explain in the following two propositions.

Proposition 5.7. *Let $k \geq 3$, and let $\bar{\psi}_k : B_k(\widehat{\Sigma}_{g,n}) \longrightarrow \ker \bar{\psi}_k$ be the homomorphism defined above. The homomorphism $\bar{\psi}_k$ lifts to an action on the level of regular coverings.*

Proof. We recall that there is an induced action of $B_n(\widehat{\Sigma}_g)$ on the homology of the covering space of $\mathbb{F}_k(\widehat{\Sigma}_{g,n})/S_k$ $\beta \in B_n(\widehat{\Sigma}_g)$ if there exists a homomorphism $\bar{\beta}_* : G_k(\widehat{\Sigma}_g) \longrightarrow G_k(\widehat{\Sigma}_g)$ such that $\bar{\beta}_* \circ \bar{\psi}_k = \bar{\psi}_k \circ \beta_*$. As we remarked previously, we may replace the homomorphism $P_k : B_k(\widehat{\Sigma}_{g,n}) \longrightarrow \ker \bar{\psi}_k$ and the group $\ker \bar{\psi}_k$ respectively by the homomorphism $\Phi_\Sigma : B_k(\widehat{\Sigma}_{g,n}) \longrightarrow G_\Sigma$ and the group G_Σ given in Section 3 of [1]. The statement follows from Lemma 3.1 of [1]. □

Proposition 5.8. *Let $k \geq 3$.*

(i) The homomorphism $\bar{\psi}_k : B_k(\widehat{\Sigma}_{g,n}) \longrightarrow \ker \bar{\psi}_k$ is a lifting extension of $p_k : B_k(\mathbb{D}_n) \longrightarrow G_k$;

(ii) let $P_k : B_k(\widehat{\Sigma}_{g,n}) \longrightarrow G_k(\widehat{\Sigma}_g)$ be a lifting extension of $p_k : B_k(\mathbb{D}_n) \longrightarrow G_k$. There is a canonical isomorphism $\iota : G_k(\widehat{\Sigma}_g) \longrightarrow \ker \bar{\psi}_k$ such that $\iota \circ P_k$ sends any generator of $B_k(\widehat{\Sigma}_{g,n})$ to the corresponding coset of $B_{k,n}(\widehat{\Sigma}_g)/\Gamma_3(B_{k,n}(\widehat{\Sigma}_g))$.

Proof. (i) In order to prove that $\bar{\psi}_k : B_k(\widehat{\Sigma}_{g,n}) \longrightarrow \ker \bar{\psi}_k$ is a lifting extension of $p_k : B_k(\mathbb{D}_n) \longrightarrow G_k$, it suffices to show that $\bar{\psi}_k(B_k(\mathbb{D}_n))$ is isomorphic to $\mathbb{Z}^2$ in $\ker \bar{\psi}_k$. This is immediate because $\bar{\psi}_k(\sigma_j) = \sigma$ for any $j = 1, \ldots, k-1$, and $\bar{\psi}_k(\zeta_l) = \zeta$ for any $l = 1, \ldots, n$. The claim then follows from Remark 5.1

(ii) Let $P_k : B_k(\widehat{\Sigma}_{g,n}) \longrightarrow G_k(\widehat{\Sigma}_g)$ be a lifting extension of $p_k : B_k(\mathbb{D}_n) \longrightarrow G_k$. Therefore $P_k(\sigma_j) = q$ for all $j = 1, \ldots, k-1$, and $P_k(\zeta_l) = t$ for all $l = 1, \ldots, n$, where $q, t \in G_k(\widehat{\Sigma}_g)$ generate a central subgroup isomorphic to $\mathbb{Z}^2$. One can check easily that these conditions imply that:

- the images of $a_1, b_1, \ldots, a_g, b_g$ commute with q and t;
- $[P_k(a_i), P_k(b_j)] = 1$ for $i \neq j$;
- $[P_k(a_i), P_k(b_i)] = q^2$.

We can therefore define a surjection $\bar{p}_k : \ker \bar{\psi}_k \longrightarrow G_k(\widehat{\Sigma}_g)$ such that $\bar{p}_k(\sigma) = q, \bar{p}_k(\zeta) = t$ and $\bar{p}_k(a_i) = P_k(a_i)$, $\bar{p}_k(b_i) = P_k(b_i)$.

We will prove that $\bar{p}_k$ is actually an isomorphism. Let $w \in \ker \bar{\psi}_k$,and consider $1 \leq r \leq g$. By Proposition 5.5, we observe that for every generator x of the presentation, we have $a_r x = x a_r$ except for $x = b_r$, in which case we have $a_r b_r = \sigma^2 b_r a_r$. Therefore, if the normal form of w is $\sigma^c \zeta^d \prod_{j=1}^{g} b_j^{n_j} a_j^{m_j}$, then $a_r w = \sigma^{2n_r} w a_r$ and $[w, a_r] = \sigma^{-2n_r}$. Similarly, $[w, b_r] = \sigma^{2m_r}$. Now if $\bar{p}_k(w) = 1$ then $\bar{p}_k([w, a_r]) = \bar{p}_k([w, b_r]) = 1$. We obtain $\bar{p}_k(\sigma^{-2n_r}) = q^{-2n_r} = 1$ and $\bar{p}_k(\sigma^{-2m_r}) = q^{-2m_r} = 1$, and thus $m_r = n_r = 0$. Since this is so for all r, we have $w = \sigma^c \zeta^d$ and $\bar{p}_k(w) = q^c t^d = 1$. Since q, t generate a subgroup that is isomorphic to $\mathbb{Z}^2$, we get $c = d = 0$ and $w = 1$. Hence $\bar{p}_k$ is injective.

□

The cases $k = 1, 2$ are still open, mainly because we do not have a finite group presentation for $\ker \bar{\psi}_k$ and $B_{k,n}(\widehat{\Sigma}_g)/\Gamma_3(B_{k,n}(\widehat{\Sigma}_g))$ in these cases. As in Proposition 5.8, there is a natural surjection from $\ker \bar{\psi}_k$ onto the group G_Σ considered in Section 3 of [1], but it is not an isomorphism.

Remark 5.9. Notice that in the proof of Proposition 5.8, we proved a slightly stronger result, i.e. that if H_k is a group for which there exists a surjection $h_k : B_k(\widehat{\Sigma}_{g,n}) \longrightarrow H_k$ satisfying $h_k(\sigma_j) = q$ for all $j = 1, \ldots, k-1$ and $h_k(\zeta_l) = t$ for all $l = 1, \ldots, n$, where the subgroup $\langle q, t \rangle$ of H_k is torsion free, then the group H_k coincides with $\ker \bar{\psi}_k$.

Using Lemma 4.9 and Theorem 4.3, one may easily adapt the proof of Proposition 5.8 in order to obtain the following result:

Proposition 5.10. *Let $k, n \geq 3$.*

(i) Let H be a group, and let $\lambda_{\widehat{\Sigma}_g} : B_n(\widehat{\Sigma}_g) \longrightarrow H$ be a surjective homomorphism such that $\lambda_{\widehat{\Sigma}_g}(B_n)$ is isomorphic to $\mathbb{Z}$. Then there is an isomorphism $\iota : H \longrightarrow B_n(\widehat{\Sigma}_g)/\Gamma_3(B_n(\widehat{\Sigma}_g))$ for which $\iota \circ \lambda_{\widehat{\Sigma}_g}$ sends every generator of $B_n(\widehat{\Sigma}_g)$ to the corresponding class of $B_n(\widehat{\Sigma}_g)/\Gamma_3(B_n(\widehat{\Sigma}_g))$.

(ii) Let H be a group. Assume that $h : B_{k,n}(\widehat{\Sigma}_g) \longrightarrow H$ is a surjective homomorphism such that the image of the subgroup generated by $\sigma_1, \ldots, \sigma_{k-1}, \tilde{\sigma}_1, \cdots, \tilde{\sigma}_{n-1}$ is isomorphic to $\mathbb{Z}^2$.

Then there is an isomorphism $\iota : H \longrightarrow \dfrac{B_{k,n}(\widehat{\Sigma}_g)}{\Gamma_3(B_{k,n}(\widehat{\Sigma}_g))}$ for which $\iota \circ h$ sends any generator of $B_{k,n}(\widehat{\Sigma}_g)$ to the corresponding coset of $\dfrac{B_{k,n}(\widehat{\Sigma}_g)}{\Gamma_3(B_{k,n}(\widehat{\Sigma}_g)}$.

The above proposition is of additional interest in the study of the lower central series of surface braid groups, since it states that the only possible extension of the length function from B_n to $B_n((\widehat{\Sigma}_g)$ is the canonical projection of $B_n(\widehat{\Sigma}_g)$ onto $B_n(\widehat{\Sigma}_g)/\Gamma_3(B_n(\widehat{\Sigma}_g))$. It is interesting to remark that in the case of closed surfaces, we have the following result:

Proposition 5.11. *Let $n \geq 3$ and let Σ be an orientable surface of positive genus. It is not possible to extend the length function $\lambda : B_n \longrightarrow \mathbb{Z}$ to $B_n(\Sigma)$. In other words there is no surjection λ_Σ of $B_n(\Sigma)$ onto a group F such that the restriction of λ_Σ to B_n coincides with λ.*

Proof. Let $\lambda_\Sigma : B_n(\Sigma) \longrightarrow F$ be such that $\lambda_\Sigma(\sigma_1) = \ldots \lambda_\Sigma(\sigma_{n-1})$. Set $\lambda_\Sigma(\sigma_1) = \sigma$. Using the group presentation of $B_n(\Sigma)$ given in [6], we see that $\sigma^{2(n+g-1)} = 1$. For further details, we refer the reader to the calculation of the group presentation of $B_n(\Sigma)/\Gamma_3(B_n(\Sigma)$ given in the proof of Theorem 1 in [8]. □

6 Appendix on exact sequences

Let us recall that the short exact sequences (MBS) and (PBS) also exist if Σ is a non-orientable surface. If $n \geq 2$ and $k \geq 2$ then the splitting of (MSB) for compact surfaces without boundary (also possibly non-orientable) is an open question. As the following example shows, the splitting of one of the two short exact sequences (SPB) and (MSB) does not imply in general that the other sequence splits.

Example 6.1. (a) Let $\Sigma = \mathbb{R}P^2$ and $n = k = 2$. First note that the pure braid sequence (SPB) does not split by [24, Theorem 3]. Secondly, using Van Buskirk's presentation of $B_n(\mathbb{R}P^2)$ [35] in the case $n = 4$, let $a = \sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\rho_1$ (which is of order 16 by [24]) and let $\Delta_4 = \sigma_1\sigma_2\sigma_3\sigma_1\sigma_2\sigma_1$

denote the 'half-twist' braid (which is of order 4 by [35]). Then [27, Proposition 15(a)] and standard properties of dicyclic groups imply that the subgroup H of $B_4(\mathbb{R}P^2)$ generated by a^2 and $a\Delta_4$ is isomorphic to the generalised quaternion group $\mathcal{Q}_{16}$ of order 16. Let $\tau_4 : B_4(\mathbb{R}P^2) \longrightarrow S_4$ denote the usual permutation homomorphism. A straightforward calculation shows that $\tau_4(H) = \langle (1,3),(2,4)\rangle$, which is isomorphic to $S_2 \times S_2$. Taking $B_{2,2}(\mathbb{R}P^2)$ to be $\tau_4^{-1}(\langle (1,3),(2,4)\rangle)$, the restriction to H of the projection $p : B_{2,2}(\mathbb{R}P^2) \longrightarrow B_2(\mathbb{R}P^2)$ given geometrically by forgetting the second and fourth strings is an isomorphism. This follows since $\ker(p)$ is torsion free and $B_2(\mathbb{R}P^2) \cong \mathcal{Q}_{16}$ [35]. In particular, p admits a section. So in this case, (MSB) splits, but (SPB) does not.

(b) Let $\Sigma = \mathbb{S}^2$. For all $n \geq 3$ and all $k \in \mathbb{N}$, (SPB) splits [17]. The question of the splitting of (MSB) is examined in [25]. For example, if $n = 3$ and $k \equiv 1 \bmod 3$, or if $n \geq 4$ and $k \not\equiv \varepsilon_1(n-1)(n-2) - \varepsilon_2 n(n-2) \bmod n(n-1)(n-2)$, where $\varepsilon_1, \varepsilon_2 \in \{0,1\}$, then (MSB) does not split. So for these values of k and n, (SPB) splits, but (MSB) does not.

Let us finish this appendix by pointing out another interesting feature when we pass from short exact sequences of classical braid groups to short exact sequences of surface braid groups. The sequence (PB) (in the case $k = 1$) has the property that the induced action of the quotient on the Abelianisation of the kernel is trivial. Following [20], we will call such a splitting extension an *almost-direct* product. The important remark for us in Theorem 3.1 of [20] is that such an exact sequences induces exact sequences on the level of the lower central series quotients (see also [28, 33]). Since P_n is an iterated almost-direct product of free groups, P_n 'inherits' various properties of F_n, and it is possible to use this structure to derive a presentation for the Lie algebra associated to the lower central series of P_n and to construct a universal finite type invariant for braid groups [33]. On the other hand, the fact that P_n acts trivially on the Abelianisation of F_n allows us to compose the Artin representation with the Magnus representation, thus yielding the Gassner representation (we refer to [2] for the details).

Proposition 6.2. *Let Σ be an orientable surface different from $\mathbb{S}^2$ and $\mathbb{T}^2$. The sequence:*

$$1 \longrightarrow P_k(\Sigma \setminus \{x_1, \ldots, x_n\}) \longrightarrow P_{k+n}(\Sigma) \longrightarrow P_n(\Sigma) \longrightarrow 1$$

defines an almost-direct product structure for $P_{k+n}(\Sigma)$ if and only if $n = 1$.

Proof. The case $n = 1$ was proved in [5]. If $n \geq 2$, as we recalled in Section 3, the sequence (PBS) splits only if Σ has boundary (Lemma 3.2). However, even when the sequence (PBS) splits, the extension is never an almost-direct product. In fact, a straightforward calculation on the level of group presentations (given for instance in [6, 8]) shows that the natural section (corresponding to adding a strand 'at infinity') does not define a trivial action on the level of Abelianisation (see also relation (v) of Proposition 4.8). The statement then follows from the

following proposition, which shows that the existence of an almost-direct product structure is independent of the choice of section. $\square$

Proposition 6.3. *Let $1 \longrightarrow K \longrightarrow G \xrightarrow{p} Q \longrightarrow 1$ be a split extension of groups. Let s, s' be sections for p, and suppose that the induced action of Q on K via s on the Abelianisation $K^{Ab} = K/[K,K]$ is trivial. Then the same is true for the section s'.*

Proof. Let $k \in K$ and $q \in Q$. By hypothesis, $s(q)\, k\, (s(q))^{-1} \equiv k \bmod [K,K]$. Let s' be another section for p. Then $p \circ s'(q) = p \circ s(q)$, and so $s'(q)\,(s(q))^{-1} \in \ker p$. Thus there exists $k' \in K$ such that $s'(q) = k'\, s(q)$, and hence

$$s'(q)\, k\, (s'(q))^{-1} \equiv k'\, s(q)\, k\, (s(q))^{-1}\, k'^{-1} \equiv k' k k'^{-1} \equiv k \bmod [K,K].$$

Thus the induced action of Q on K^{Ab} via s' is also trivial. $\square$

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